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DYNAMICS OF HEATING A SOLID BODY WITH THERMAL DESTRUCTION OF ITS SURFACE

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Based on the laws governing thermal destruction and on differential transformations, a solution of the equation of heat conduction with mass entraintment has been obtained. Analytical expressions accounting for wave disturbance of the temperature field are suggested. Using an alloyed quartz glazed ceramics as an example, a good agreement between the predicted and experimental temperature fields in a wide range of mass entrainment velocities and times of heating is shown.

Investigations of the interaction of a solid body with a high-temperature medium are of great interest for studying various technological processes and their optimization. The influence of concentrated energy fluxes is typical of the processes of cutting, melting, material surface finish as well as of the use of thermal shielding of rocket-space equipment.

At the present time fairly many mathematical models have been developed that allow one to estimate the heating and thermal destruction of the materials exposed to intense heat fluxes [1, 2]. However, usually these models do not allow for the regularities and parameters established in numerical-experimental investigations of thermoprotective materials and therefore they do not always agree with the results of experimental investigations, especially for low thermal-conductivity materials.

In [3], the following regularities and parameters were generalized that had been established on heating and thermal destruction of the surface of thermoprotective materials: a) the relationship governing the progress of nonstationary heating of material with material entrainment from the surface; b) the relationship governing the establishment of a stationary regime of mass entrainment; c) the parameter of nonstationary entrainment d_0 weakly dependent on the conditions of heating and on all the properties of the material except for its thermal conductivity; d) the constant of thermal destruction K_{T_d} that determines the laws of heating, changes in the surface temperature, and in the velocity of mass entrainment in the regime of nonstationary destruction of the material surface; e) the relationship governing the attainment of the limiting energy capacity of internal heat absorption processes which is attainable upon equality of the thicknesses of the heated and entrained layers of the material (Fig. 1).

The latter relationship can be illustrated with the example of a spring. Let us imagine that at the time of action of an intense thermal load the compressed coils of the spring are the isotherms of a temperature field. If we allow the spring to expand, then at the moment of its full expansion the coil that was farther away from the middle of the spring will cover a greater path. With a decrease in the temperature of the isotherm, the velocity of its movement increases, and by the time of the establishment of a stationary regime of heating (a fully expanded spring) the path covered by the isotherm that bounds the heated layer will be the larger the lower its temperature. Precisely at this time the heated layer attains the maximum thickness and the quantity of heat accumulated in it attains its limiting value. Naturally the velocity of movement of all the isotherms whose temperature is higher than that of the isotherm that bounds the heated layer is equal to the rate of destruction of the heated surface. The fact that virtually all isotherms of the temperature field cease to "run away" from the destroying surface, when the depth of their occurrence attains the thickness of the entrained layer, is also clearly seen on the example of a spring. Since the spring expands to both sides (to the sides of low and high temperatures) equally, one-half of it that arrives in the region of the temperatures of destruction burns out similarly to the entrained layer.

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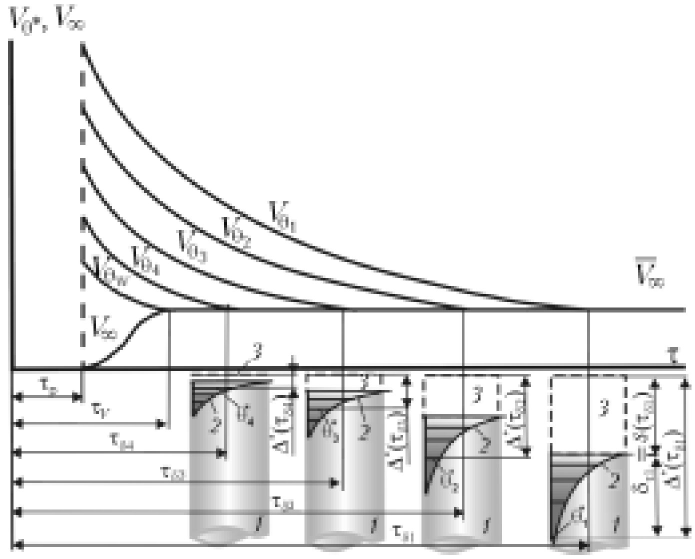

Fig. 1. Model of heating and entrainment of thermoprotective material: 1) specimen; 2) temperature profile; 3) entrained layer; θ_1^*–θ_4^*, isotherms bounding the heated layer whose thickness is equal to the entrained layer; $\Delta^*(\tau_{\delta 1}) = \delta_{\tau 1} + S(\tau_{\delta 1})$, overall thickness of the heated, $\delta_{\tau 1}$, and entrained, $S(\tau_{\delta 1})$, layers at time $\tau_{\delta 1}$; $V_{\theta \text{w}}$, $V_{\theta 1}$–$V_{\theta 4}$, velocities of movement of isotherms with surface temperatures θ_w and θ_1^*–θ_4^*, respectively; $\tau_{\delta 1}$–$\tau_{\delta 4}$, times of establishment of the stationary velocity of the movement of isotherms with dimensionless temperatures θ_1^*–θ_4^*, respectively; $\theta_1^* < \theta_2^* < \theta_3^* < \theta_4^* < \theta_\text{w}^* = 1$.

In [4], a wave disturbance of the temperature field was established; it sets in when the supplied heat flux exceeds the level needed for heating without destruction of the material surface. This disturbance is attenuated on increase in the time of heating if there is no mass entrainment and shortens the time of attainment of the stationary regime of surface destruction with mass entrainment. However, the analytical relations suggested did not ensure agreement between the calculated results and experimental data in the entire range of velocities and times of heating considered. For example, the approximation obtained to calculate the temperature field in the melting of the material surface without removal of the melt film showed that the disturbance of the temperature field damps out more rapidly than follows from the experiment.

In [5], equations for calculation of the temperature field were suggested:

at $S(\tau) < y < y_\text{s}$ (stationary section)

$$T_1(y, \tau) = T_0 + (\overline{T}_\text{w} - T_0) \exp\left[-\frac{\overline{V}_\infty}{a}(y - \overline{V}_\infty \tau + d_0)\right], \tag{1}$$

at $y \ge y_\text{s}$ (nonstationary section)

$$T_2(y, \tau) = T_0 + (\overline{T}_\text{w} - T_0) \exp\left(-\frac{\overline{V}_\infty}{a}[y - S(\tau)]\right)\left[1 - \text{erf}\left(\frac{y - y_\text{s}}{2\sqrt{a\tau}}\right)\right]. \tag{2}$$

Here the position of the boundary y_s is determined by the equality between the thicknesses of the heated δ_T and entrained $S(\tau)$ layers of the material (Fig. 1). Thus, $y_\text{s} \approx S(\tau) + \delta_T \approx 2S(\tau)$ is the coordinate which is reckoned from the original surface and which determines the lower boundary of the stationary heated layer.

We will show that (1) and (2) satisfy the linear heat conduction equation:

$$\frac{\partial T}{\partial \tau} = a \frac{\partial^2 T}{\partial y^2} \tag{3}$$

under the following initial and boundary conditions:

$$1) -\left(\lambda \frac{\partial T}{\partial y}\right)\bigg|_{y=S(\tau)} = \rho \overline{V}_{\infty} c\,(\overline{T}_{\mathrm{w}} - T_0) = \rho \overline{V}_{\infty} H\,(T_{\mathrm{w}}) = \mathrm{const}\,, \quad 2)\; T\,(y, \tau)\,\big|_{\,y=S(\tau)} = \overline{T}_{\mathrm{w}}\,,$$

$$3)\text{ for }\;\tau > 0\;\text{ and }\; y \to \infty \quad T \to T_0\,, \quad 4)\; y > S\,(\tau)\,. \tag{4}$$

To solve (3) under boundary conditions (4) we shall resort to the symbolic method [6] based on G. E. Petukhov's differential transformations [7]. The essence of the method is as follows. We will consider the physical processes which are described by the function $u(x_1, x_2)$ of two independent variables in the region:

$$0 \le |x_1| \le H_1 \le \infty\,, \tag{5}$$

$$0 \le |x_2| \le H_2 \le \infty\,. \tag{6}$$

We will model the processes of the form of $u(x_1, x_2)$ by introducing a system of two one-dimensional differential transformations of the form

$$U\,(k_1, x_2) = \frac{H_1^{k_1}}{k_1!}\left(\frac{\partial^{k_1} u\,(x_1, x_2)}{\partial x_1^{k_1}}\right)_{x_1=0}\,, \tag{7}$$

$$u\,(x_1, x_2) = \sum_{k_1=0}^{\infty}\left(\frac{x_1}{H_1}\right)^{k_1} U\,(k_1, x_2)\,, \tag{8}$$

$$U\,(x_1, k_2) = \frac{H_2^{k_2}}{k_2!}\left(\frac{\partial^{k_2} u\,(x_1, x_2)}{\partial x_2^{k_2}}\right)_{x_2=0}\,, \tag{9}$$

$$u\,(x_1, x_2) = \sum_{k_2=0}^{\infty}\left(\frac{x_2}{H_2}\right)^{k_2} U\,(x_1, k_2)\,, \tag{10}$$

where the integral arguments k_1 and k_2 take the values 0, 1, 2, 3, ... Expression (7) describes direct differential transformations of the function $u(x_1, x_2)$ into the function $U(k_1, x_2)$ of the integral argument k_1 and of the independent variable x_2, which is called a transform or a differential spectrum of the process $u(x_1, x_2)$. The inverse differential transformations (8) allow one to reconstruct the process $u(x_1, x_2)$ in the region of inverse transforms by using the differential spectrum $U(k_1, x_2)$. Similarly, expressions (9) and (10) describe respectively direct and inverse differential transformations with respect to the variable x_2.

The differential spectra $U(k_1, x_2)$ and $U(x_1, k_2)$ are the analogs in the region of transforms of a physical process whose mathematical model is described by the function $u(x_1, x_2)$. In order to model physical processes one can use the differential spectrum $U(k_1, x_2)$ or $U(x_1, k_2)$. In some of the boundary-value problems the necessity for studying both of them appears.

The basic properties of one-dimensional differential transformations established in [7] are valid for both forms of transformations (7) and (9). Basic mathematical operations in the region of transforms (7) and (9) are performed by the rules of correspondence which are defined by the following expressions:

$$u(x_1, x_2) \pm v(x_1, x_2) \Leftrightarrow \begin{cases} U(k_1, x_2) \pm V(k_1, x_2), \\ U(x_1, k_2) \pm V(x_1, k_2), \end{cases} \tag{11}$$

$$Cu(x_1, x_2) \Leftrightarrow \begin{cases} CU(k_1, x_2), \\ CU(x_1, k_2), \end{cases} \tag{12}$$

$$u(x_1, x_2)\, v(x_1, x_2) \Leftrightarrow \begin{cases} U(k_1, x_2) * V(k_1, x_2), \\ U(x_1, k_2) * V(x_1, k_2), \end{cases} \tag{13}$$

$$\frac{\partial^m u(x_1, x_2)}{\partial x_1^m} \Leftrightarrow D_1^m U(k_1, x_2), \tag{14}$$

$$\frac{\partial^m u(x_1, x_2)}{\partial x_2^m} \Leftrightarrow D_2^m U(x_1, k_2), \tag{15}$$

where

$$U(k_1, x_2) * V(k_1, x_2) = \sum_{l=0}^{l=k_1} U(l, x_2)\, V(k_1 - l, x_2); \tag{16}$$

$$U(x_1, k_2) * V(x_1, k_2) = \sum_{l=0}^{l=k_2} U(x_1, l)\, V(x_1, k_2 - l); \tag{17}$$

$$D_1^m U(k_1, x_2) = \frac{(k_1 + m)!}{k_1! H_1^m} U(k_1 + m, x_2); \tag{18}$$

$$D_2^m U(x_1, k_2) = \frac{(k_2 + m)!}{k_2! H_2^m} U(x_1, k_2 + m). \tag{19}$$

In (11)–(13) the mathematical operations in the region of transforms are denoted by curly brackets. The upper line of the expression in the curly brackets denotes performing the operations in the region of transforms (7), whereas the lower one — in the region of transforms (9).

Expression (11) shows that to the operations of summation and subtraction in the region of inverse transforms there correspond the same operations of summation and subtraction of differential spectra in the region of transforms (7) and (9). To the multiplication of the function $u(x_1, x_2)$ by the constant C (12) there corresponds multiplication of the differential spectra $U(k_1, x_2)$ and $U(x_1, k_2)$ by the same constant. To the operation of multiplication of two functions in the region of inverse transforms (13) there corresponds a special operation of multiplication (symbol *) of two differential spectra in the region of transforms (7) and (9). The operation of the m-fold differentiation of the function $u(x_1, x_2)$ with respect to the variable x_1 in the region of transforms (7) is denoted by the symbol D_1^m in (14). Similarly the symbol D_2^m in (15) denotes the m-fold differentiation of the function $u(x_1, x_2)$ with respect to the variable x_2 in the region of transforms (9).

The operation of multiplication * of two differential spectra in the region of transforms (7) and (9) is resolved by expressions (16) and (17), respectively. The operation of the m-fold differentiation in the regions of transforms (7) and (9) is realized by expressions (18) and (19), respectively.

We will consider a mathematical model of a physical process in the form of a partial differential equation with two independent variables:

$$f\left(x_1, x_2, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_2^2}\right) = 0 . \tag{20}$$

Equation (20) may have an infinite number of particular solutions. Modeling of specific physical processes requires that of all the solutions of Eq. (20) we should select that which satisfies the boundary conditions. As a rule, the boundary conditions are given on the boundary Γ of the medium in which a physical process proceeds:

the Dirichlet condition

$$u(x)\big|_{x \in \Gamma} = \psi(x) , \tag{21}$$

the Neumann condition

$$\left.\frac{du(x)}{dv}\right|_{x \in \Gamma} = \psi(x) , \tag{22}$$

a mixed condition

$$\frac{du(x)}{dv} + \beta u = \psi(x) , \tag{23}$$

where $du(x)/dv$ is the derivative taken at the point of the surface Γ in the direction of the normal to it.

We will shift Eq. (20) into the region of transforms by means of differential transformations (7) and (9) using the rules of correspondence (11)–(19). As a result, we obtain two transforms:

$$F_1\left[k_1, x_2, U(k_1, x_2), D_1 U(k_1, x_2), \frac{dU(k_1, x_2)}{dx_2}, D_1 \frac{dU(k_1, x_2)}{dx_2}, D_1^2 U(k_1, x_2), \frac{d^2 U(k_1, x_2)}{dx_2^2}\right] = 0 , \tag{24}$$

$$F_2\left[x_1, k_2, U(x_1, k_2), \frac{dU(x_1, k_2)}{dx_1}, D_2 U(x_1, k_2), D_2 \frac{dU(x_1, k_2)}{dx_1}, \frac{d^2 U(x_1, k_2)}{dx_2^2}, D_2^2 U(x_1, k_2), \right] = 0 , \tag{25}$$

where F_1 and F_2 are the transforms of the function f that were obtained by means of transformations (7) and (9), respectively. Equations (24) and (25) are ordinary differential equations, and each of them is an analog of Eq. (20) in the region of transforms.

Boundary conditions (21) and (22) are transferred into the regions of transforms (7), (9) or are expressed in terms of the differential spectra $U(k_1, x_2)$ and $U(x_1, k_2)$ as follows:

$$u(0, x_2) = U(0, x_2) , \quad u(x_1, 0) = U(x_1, 0) , \tag{26}$$

$$u(H_1, x_2) = \sum_{k_1=0}^{\infty} U(k_1, x_2) , \tag{27}$$

$$u(x_1, H_2) = \sum_{k_2=0}^{\infty} U(x_1, k_2) , \tag{28}$$

$$U(1, x_2) = H_1 \left[\frac{\partial u(x_1, x_2)}{\partial x_1} \right]_{x_1=0} , \tag{29}$$

$$U(x_1, 1) = H_2 \left[\frac{\partial u(x_1, x_2)}{\partial x_2} \right]_{x_2=0} , \tag{30}$$

$$\left[\frac{\partial u(x_1, x_2)}{\partial x_1} \right]_{x_1=H_1} = \frac{1}{H_1} \sum_{k_1=0}^{\infty} (k_1 + 1)\, U(k_1 + 1, x_2) , \tag{31}$$

$$\left[\frac{\partial u(x_1, x_2)}{\partial x_2} \right]_{x_2=H_2} = \frac{1}{H_2} \sum_{k_2=0}^{\infty} (k_2 + 1)\, U(x_1, k_2 + 1) . \tag{32}$$

Boundary conditions (23) are expressed in terms of discrete values of the differential spectra on the basis of expressions (26)–(32). Generally the problem of the solution of partial differential equation (20) with boundary conditions (21)–(23) was reduced to the solution of ordinary differential equations (24) and (25) in the region of transforms for the differential spectra $U(k_1, x_2)$ and $U(x_1, k_2)$ at boundary conditions (26)–(32). This problem can be solved by the well-known methods. In the case of linear differential equations (24) and (25), one can apply general methods of solution of ordinary differential equations, methods based on integral transformations, or methods based on differential transformations suggested in [7]. In the case of nonlinear differential equations (24) and (25) one should recommend their solution in an analytical and numerical-analytical form on the basis of differential transformations [7].

When the problem was solved in the region of transforms, conversion of the transform of the solution of problem (20)–(23) into the region of inverse transforms is made. This conversion can be made in different ways.

The first technique consists of using the tables of correspondence between transforms and inverse transforms analogous to the correspondence tables compiled for integral transformations. The tables of correspondence of transforms and inverse transforms for one-dimensional differential transformations are given in [7].

The second technique employs the reconstruction of the function $u(x_1, x_2)$ by the differential spectra $U(k_1, x_2)$ and $U(x_1, k_2)$ according to inverse transformations (8) and (10).

The third technique consists of transition to the region of the transforms of integral Laplace transformations by the expressions [7]:

$$U(p, x_2) = \sum_{k_1=0}^{\infty} \frac{k_1!}{p^{k_1+1}} \frac{U(k_1, x_2)}{H_1^{k_1}} , \quad U(x_1, p) = \sum_{k_2=0}^{\infty} \frac{k_2!}{p^{k_2+1}} \frac{U(x_1, k_2)}{H_2^{k_2}} .$$

After the passage into the region of Laplace transforms, one can use methods of transition to the inverse transform that were developed for integral Laplace transformations.

The fourth technique is based on the method of the balance of differential spectra suggested in [7]; it consists of the choice of the form of an approximating function with free coefficients or a series in the well-known basis functions converting them by differential transformations (7) and (9) into the region of transforms and of equating the discretes of these transforms into the differential spectra $U(k_1, x_2)$ and $U(x_1, k_2)$ of transforms (24) and (25) at the same values of the integral arguments k_1 and k_2. This results in a system of finite equations for determination of the coefficients of the series or free coefficients of the approximating function.

We will note the characteristic features of the proposed symbolic method [6]. It is an exact operational method, and provided problem (20)–(23) is solved analytically, it yields its exact analytical solution. According to (5) and (6), the proposed method admits a change in variables within both finite and infinite limits. Rather simple expressions (16) and (17) for the operation of multiplication to be performed in the region of transforms allow one to extend the symbolic method to nonlinear boundary-value problems.

Differential transformations (7)–(10) offer wide possibilities of performing analytical and numerical-analytical transitions from the region of transforms to inverse transforms. In many practical cases the boundary-value problem (20)–(23) is solved on the basis of one transform (24) or (25).

For the solution of the heat-conduction equation (3) under boundary conditions (4) we shall use the above-described symbolic method based on differential transformations (7). Having denoted $x_1 = y$; $x_2 = \tau$; $H_1 = H_y$; $U(k_1, x_2) = T(q, \tau)$; $k_1 = q$; $u(x_1, x_2) = T(y, \tau)$, we will write

$$T(q, \tau) = \frac{H_y^q}{q!} \left[\frac{\partial^q T(y, \tau)}{\partial y^q} \right]_{y=S(\tau)} ,$$

where q is the integral argument that takes the values 0, 1, 2, 3, ..., ∞. The use of differential transformations converts the partial differential equation (3) in the region of transforms into the ordinary differential equation

$$\frac{dT(q, \tau)}{d\tau} - aD_y^2 T(q, \tau) = 0 ,$$

whose general solution yields the expression

$$T(q, \tau) = C(q) \exp(\tau a D_y^2) ,$$

where $C(q)$ is the unknown function of the integral argument q to be determined from boundary conditions (4).

We will represent the first boundary condition of (4) in the form

$$\left. \frac{\partial T(y, \tau)}{\partial y} \right|_{y=S(\tau)} + hT(y, \tau) \big|_{y=S(\tau)} = hT_0 , \quad h = \bar{V}_\infty / a ,$$

it is satisfied by the function

$$T(y, \tau) = C_1 \exp[-h(y - S(\tau))] + T_0 , \quad C_1 = \text{const} .$$

For the second boundary condition of (4) we may write the expression

$$T(y, \tau) \big|_{y=S(\tau)} = \bar{T}_{\text{w}} = C_1 + T_0 , \quad C_1 = \bar{T}_{\text{w}} - T_0 .$$

Consequently, boundary conditions (4) are satisfied by the function

$$T(y, \tau) = T_0 + (\bar{T}_{\text{w}} - T_0) \exp[-h(y - S(\tau))] ,$$

which at $\tau = 0$ is transformed as

$$T(y, 0) = T_0 + (\bar{T}_w - T_0) \exp(-hy) .$$

We will put the latter expression into the region of differential transformations:

$$T(q, 0) = T_0 \sigma(q) + (\bar{T}_w - T_0) E(q) , \quad \sigma(q) = \begin{cases} 1, & q = 0 , \\ 0 , & q \neq 0 . \end{cases}$$

Equating solution (3) in the region of transforms at $\tau = 0$ to the transform of the function which satisfies boundary conditions (4), we will write an equation for determining the unknown function $C(q)$:

$$T(q, 0) = C(q) = T_0 \sigma(q) + (\bar{T}_w - T_0) E(q) .$$

As a result, we obtain the solution of (3) at boundary conditions (4) in the region of transforms:

$$T(q, \tau) = \left[T_0 \sigma(q) + (\bar{T}_w - T_0) E(q) \right] \exp(\tau a D_y^2) .$$

This expression should be converted from the region of transforms into the region of inverse transforms to find the sought solution in the form of $T(y, \tau)$. For this purpose, we shall expand the exponent into a power series:

$$T(q, \tau) = T_0 \left[\sigma(q) + \frac{a\tau}{1!} D_y^2 \sigma(q) + \frac{(a\tau)^2}{2!} D_y^4 \sigma(q) + ... \right]$$

$$+ (\bar{T}_w - T_0) \left[E(q) + \frac{a\tau}{1!} D_y^2 E(q) + \frac{(a\tau)^2}{2!} D_y^4 E(q) + ... \right] .$$

We shall transfer it into the region of inverse transforms in which all terms $D_y^m \sigma(q)$ vanish, since differentiation of any order of a constant function is equal to zero. For the terms of the form $D_y^m \sigma(q)$ in the region of inverse transforms the following expressions are adequate;

$$\frac{d^{2n} \exp(-hy)}{dy^{2n}} = (h^2)^n \exp(-hy) , \quad m = 2n .$$

To the transform $\sigma(q)$ in the region of inverse transforms there corresponds unity. With allowance for the indicated relations, the solution obtained in the region of inverse transforms has the form

$$T(y, \tau) = T_0 + (\bar{T}_w - T_0) \exp(-hy) \left[1 + \frac{ah^2}{1!} \tau + \frac{(ah^2)^2}{2!} \tau^2 + ... \right] .$$

The expression in the square brackets is the power series expansion of the function $\exp(ah^2\tau)$ with allowance for which the solution in the region of inverse transforms becomes

$$T_1(y, \tau) = T_0 + (\bar{T}_w - T_0) \exp(-hy + h^2 a\tau) . \tag{33}$$

The obtained solution exactly satisfies the heat-conduction equation (3) and boundary conditions (4). Since, according to [3], the entrained layer is equal to $S(\tau) = \underline{V}_\infty \tau - d_0$, Eq. (33) goes over into Eq. (1) which can be used when $\tau > \tau_v$ and $S(\tau) < y < y_s$.

To obtain the solution of (3) over the section $y \geq y_s$, we shall avail ourselves of shifted differential transformations. In this case, the shifted variables will have the form

$$y_{sh} = 1 - 2S(\tau) \quad \text{and} \quad T_{sh}(y_{sh}, \tau) = T(y_{sh}, \tau) - T_0 . \tag{34}$$

Equation (33) should be used as a boundary condition at $y = y_s$ due to the fact that $S(\tau) \approx \overline{V}_\infty \tau$.

In shifted variables the boundary-value problem is formulated as follows: Find the solution of the equation

$$\frac{\partial T_{sh}(y_{sh}, \tau)}{\partial \tau} = a \frac{\partial^2 T_{sh}(y_{sh}, \tau)}{\partial y_{sh}^2}$$

at boundary conditions

$$T_{sh}(y_{sh}, \tau)\big|_{y_{sh}=0} = (\overline{T}_w - T_0) \exp\left(-\frac{\overline{V}_\infty^2}{a}\tau\right), \quad \lim_{y_{sh}\to\infty} T_{sh}(y_{sh}, \tau) = 0 . \tag{35}$$

We will solve the set problem by differential transformations (9) with respect to the time argument τ at $x_1 = y_{sh}$, $x_2 = \tau$, $H_2 = H_\tau$, $k_2 = k$, $U(x_1, k_2) = T(y_{sh}, k)$, $u(x_1, x_2) = T_{sh}(y_{sh}, \tau)$:

$$T_{sh}(y_{sh}, k) = \frac{H_\tau^k}{k!}\left[\frac{\partial^k T_{sh}(y_{sh}, \tau)}{\partial \tau^k}\right]_{\tau=0},$$

where k is the integral argument that takes the values 0, 1, 2, 3, ... ∞.

The use of differential transformations of this form converts the partial differential equation in the region of transformations into an ordinary differential equation:

$$\frac{d^2 T_{sh}(y_{sh}, k)}{dy_{sh}^2} - \frac{1}{a} D T_{sh}(y_{sh}, k) = 0 ,$$

whose general solution has the form

$$T_{sh}(y_{sh}, k) = A(k) \exp\left(-\frac{y_{sh}}{\sqrt{a}}\sqrt{D}\right) + B(k) \exp\left(-\frac{y_{sh}}{\sqrt{a}}\sqrt{D}\right),$$

where $A(k)$ and $B(k)$ are the unknown functions of the integral argument k to be determined from the given boundary conditions.

We will transfer the boundary condition (35) into the region of transforms:

$$\lim_{y_{sh}\to\infty} T_{sh}(y_{sh}, k) = 0 .$$

If in $T_{sh}(y_{sh}, k)$ we assume that $B(k) = 0$ and substitute it into the second boundary condition, we will obtain

$$\lim_{y_{sh}\to\infty} A(k) \exp\left(-\frac{y_{sh}}{\sqrt{a}}\sqrt{D}\right) = 0 .$$

Thus, the second boundary condition is satisfied if the solution of the boundary-value problem in the region of inverse transforms has the form

$$T_{sh}(y_{sh}, k) = A(k) \exp\left(-\frac{y_{sh}}{\sqrt{a}}\sqrt{D}\right).$$

We shall put the boundary condition (35) into the region of differential transformations:

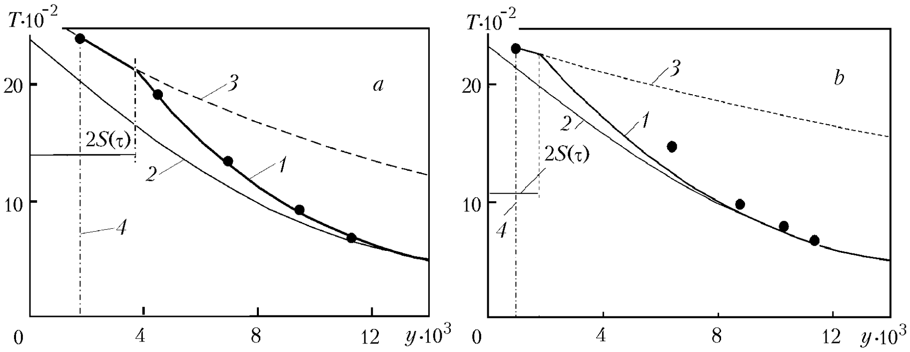

Fig. 2. Comparison between predicted and experimental temperature profiles in alloyed quartz glass ceramics: (QGC): a) $\bar{V}_\infty = 0.05{\cdot}10^{-3}$ m/sec, $\bar{T}_w = 2400$ K, $a = 0.65{\cdot}10^{-6}$ m^2/sec (at the 50th second of heating); b) $\bar{V}_\infty = 0.02{\cdot}10^{-3}$ m/sec, $T_w = 2350$ K, $a = 0.65{\cdot}10^{-6}$ m^2/sec (at the 60th second of heating); 1) calculation by Eq. (1) at $S(\tau) < y \le y_s$ and by Eq. (2) at $y \ge y_s$; 2) by Eq. (36); 3) by Eq. (1); 4) position of the heated surface; points — experiment. T, K; y, m.

$$T_{sh}\,(y_{sh}, k) = A\,(k) \exp\left(-\frac{y_{sh}}{\sqrt{a}}\sqrt{D}\right).$$

Equating the first boundary condition in the region of transforms to the solution of the boundary-value problem at $y_{sh} = 0$, we obtain an expression for the unknown function $A(k)$ in the form

$$A\,(k) = (\bar{T}_w - T_0)\,E\,(k)\,.$$

The substitution of this expression into the solution of the boundary-value problem at $E(k) = \sigma(k) * E(k)$, where the symbol * denotes the operation of multiplication in the region of transforms, yields the sought solution in the region of transforms:

$$A\,(k) = (\bar{T}_w - T_0)\,E\,(k)\,.$$

We transfer the obtained solution of the boundary-value problem to the region of inverse transforms. For this purpose we introduce the notation

$$\Phi\,(k) = \exp\left(-\frac{y_{sh}}{\sqrt{a}}\sqrt{D}\right)\sigma\,(k)\,.$$

According to the table of correspondence of transforms and inverse transforms given in [7], to this function in the region of inverse transforms there corresponds the expression $1 - \text{erf}\left(\frac{y_{sh}}{2\sqrt{a\tau}}\right)$.

With allowance for the notation introduced for $\Phi(k)$, the solution of the boundary-value problem in the region of transforms takes the form

$$T_{sh}\,(y_{sh}, k) = (\bar{T}_w - T_0)\,[\Phi\,(k) * E\,(k)]\,.$$

Since to the symbol * in the region the of inverse transforms there corresponds an ordinary operation of multiplication and to the transform $E(k)$ there corresponds the inverse transform $\exp\left(-\frac{\bar{V}_\infty^2}{a}\tau\right)$, we find the solution of the boundary-value problem in the region of inverse transforms:

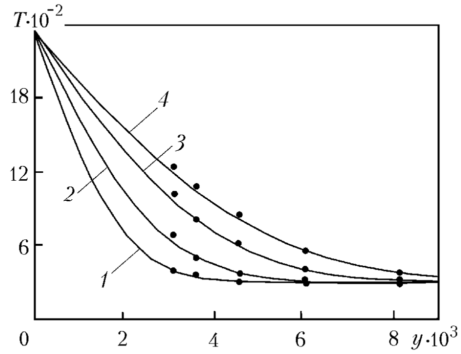

Fig. 3. Comparison between predicted and experimental temperature profiles in alloyed QGC at heating times $\tau \le \tau_T$: 1–4) calculation by Eq. (36) at the 2nd, 4th, 8th, and the 13th seconds of heating at T_w = const; points — experiment (q_c = 2700 kW/m^2, T_w = 2350 K, $a = 0.6{\cdot}10^{-6}$ m^2/sec, τ_T = 13 sec). T, K; y, m.

$$T_{sh}(y_{sh}, \tau) = (\overline{T}_w - T_0) \exp\left(-\frac{\overline{V}_\infty^2}{a}\tau\right)\left[1 - \operatorname{erf}\left(\frac{y_{sh}}{2\sqrt{a\tau}}\right)\right].$$

With allowance for Eq. (34) we obtain

$$T_2(y, \tau) = T_0 + (\overline{T}_w - T_0) \exp\left(-\frac{\overline{V}_\infty^2}{a}\tau\right)\left[1 - \operatorname{erf}\left(\frac{y - y_s}{2\sqrt{a\tau}}\right)\right].$$

This solution can be rewritten in the form of Eq. (2) if we take into account that the coordinate of the mobile surface is determined, on the one hand, by the entrained layer $y' = S(\tau) \approx \overline{V}_\infty\tau$ and, on the other hand, is expressed in terms of the immobile coordinate y as $y' = y - S(\tau)$.

Thus, Eqs. (1) and (2) satisfy the heat-conduction equation (3) and experimental data at mass entrainment velocities exceeding $0.04{\cdot}10^{-3}$ m/sec (Fig. 2a). However, at lower rates of surface destruction the wave disturbance of the temperature field found in [4] manifests itself more strongly, and it seems that for this reason the calculations done by Eq. (2) differ appreciably from the experiment (Fig. 2b). Moreover, at the boundary $y_s \approx 2S(\tau)$ expression (2) does not ensure a smooth transition to the stationary region $S(\tau) < y < y_s$ where Eq. (1) is used for calculations.

It is known [8] that the solution of (3) for a half-space under boundary-value conditions of the first kind T_w = const that presume instantaneous establishment of temperature on the surface has the form

$$\theta^* = \frac{T^* - T_0}{\overline{T}_w - T_0} = \operatorname{erfc}\left(\frac{y}{2\sqrt{a\tau}}\right) = \operatorname{erfc}\left(\frac{K}{2}\right), \tag{36}$$

where the depth of the heated layer bounded by the isotherm with a dimensionless temperature θ^* obeys the relation

$$y \approx K\sqrt{a\tau}\,. \tag{37}$$

However, independently of the time of attainment of a constant temperature on the surface (for some experiments it exceeded 10 sec) the temperature of the heated material in all of the cases considered was even higher than the values obtained in calculation by Eq. (36) (Fig. 3).

From this it can be concluded that under the conditions where the constant heat flux supplied exceeds the level needed for heating without the material surface destruction, the rate of its heating cannot be smaller than the value determined by the condition of instantaneous establishment of T_w = const. Since in some cases the experimentally established rate of heating was higher than that prescribed by the condition T_w = const, it seems that on instantaneous supply of high heat fluxes to the material surface some thermal dynamic effect appears which causes wave disturbance

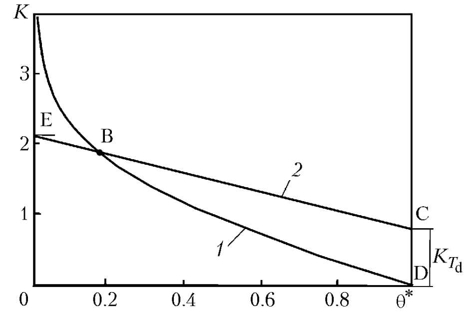

Fig. 4. Dependence of K on θ^*: 1) at the material surface temperature lower than T_d, calculation by Eq. (36); 2) at the material surface temperature higher than T_d and with mass entrainment, calculation by Eq. (38); the values of K at the points C, B, E — 0.74, 1.83, and 2.09, respectively.

of the temperature field. At the same time the well-known solutions of heat-conduction equations show that the appearance of temperature waves is possible either under the corresponding boundary conditions (for example, [9]) or with account for the finite velocity of heat propagation when the hyperbolic heat-conduction equation is used [10].

In [3], it is shown that with a change in the material surface temperature from the beginning of destruction (melting) T_d up to the steady-state value T_w, in the surface layer of the material a certain quantity of heat is accumulated determined by the constant K_{T_d}. According to the diagram of thermal destruction of a material [11] (Fig. 4), this heat propagates in the zone determined by the point of intersection of dependences (36) and

$$K = -\frac{1}{K_{T_d}}\theta^* + \frac{K_{T_d}^2}{1 - K_{T_d}} \tag{38}$$

(from the latter dependence at $\theta^* = 1$ the value of K_{T_d} equal to ~0.74 was obtained). The coordinate of this point at the rates of surface destruction for homogeneous materials of less than $0.05 \cdot 10^{-3}$ m/sec corresponds to the values $\theta^* \approx 0.2$ and $K \approx 1.83$. The path covered by the isotherm with a temperature $\theta^* > 0.2$ in a time longer than τ_T and reckoned from the initial surface is determined by an equation analogous to Eq. (37):

$$y = K\sqrt{a}\,(\sqrt{\tau} - \sqrt{\tau_\xi})\,. \tag{39}$$

Here the time τ_ξ is determined by the point of intersection of the linear dependence (39) with the abscissa axis and, according to experimental data, is equal to ~1 sec [3].

In the case of surface melting without mass entrainment in the time of the establishment of the surface temperature τ_T the coefficient K at $\theta^* = 1$ also attains the value of K_{T_d} (Fig. 4). However, the value of K at $\theta^* \approx 0.2$ must be smaller than 1.83 [4].

The results obtained allow the assumption that precisely in the zone bounded by $\theta^* \approx 0.2$ (the DBC portion in Fig. 4) the heat accumulated in the surface layer of the material propagates in time τ_T. The position of the lower boundary of such a zone can be found from (9) or (11), if we determine the characteristic time τ. In the case of surface melting without removal of the melt film in such a time we may evidently take τ_T, since at this instant of time the quantity of heat accumulated in the surface layer of the material, and probably one of the reasons for the wave disturbance, attains a maximum value. Thereafter the accumulated heat propagates in the heated layer due to the thermal conductivity of the material, and the wave disturbance damps out.

With mass entrainment from the surface of homogeneous materials, the accumulated heat is removed by the entrained layer $S(\tau_v)$ in the time of establishment of the stationary velocity of mass entrainment τ_v. The value of $S(\tau_v)$ and τ_v are determined with the aid of the constant K_{T_d} and of the parameter of nonstationary mass entrainment d_0 [3] by the formulas

$$S(\tau_v) = \frac{d_0}{K^2_{T_d}} \quad \text{and} \quad \tau_v = \frac{K^2_{T_d} + 1}{K^2_{T_d}} \frac{d_0}{\overline{V}_\infty}.$$

In this case the wave disturbance of the temperature field accelerates the attainment of a stationary regime of heating. The region of the temperature field in which this disturbance propagates in time τ_v can be evaluated by (39).

In [11], it is shown that for transition from the coefficients on the energy diagram, which characterizes the heating and destruction of a material, to thermal energy they must be multiplied by $\sqrt{\pi}$. To characterize the wave disturbance of the temperature field in surface melting without mass entrainment we will introduce the parameter

$$\alpha_T = \pi / y_T\,, \quad y_T = K \sqrt{a\tau_T}\,, \tag{40}$$

and with mass entrainment

$$\alpha_v = \pi / y_v\,, \quad y_v = K \sqrt{a}\,(\sqrt{\tau_v} - 1)\,. \tag{41}$$

The results obtained in [4, 5] show that the distribution of temperature in the material, both with mass entrainment and without it, tends to base equations (1), (2), (36) which satisfy the heat-conduction equation. Therefore the problem is reduced to finding additional terms for these equations which would take into account the wave disturbance of the temperature field. Let us assume that with mass entrainment this additional term has a form similar to (2):

$$\Delta T(y, \tau) = A\,(\overline{T}_w - T_0) \exp\left(-\alpha_v [y - S(\tau)]\right) \operatorname{erf}\left(\frac{y - y_s}{2\sqrt{a\tau}}\right), \tag{42}$$

where instead of the parameter $h = \overline{V}_\infty / a$, which accounts for heat transfer with the velocity of movement of the destruction surface, the parameter α_v (41) is introduced, which characterizes the region of propagation of the wave disturbance of the temperature field in time τ_v.

The coefficient A is determined from the boundary conditions of the fourth kind at the point $y = y_s$

$$\lambda \frac{\partial T_1(y, \tau)}{\partial y}\bigg|_{y=y_s} = \lambda \frac{\partial \Delta T(y, \tau)}{\partial y}\bigg|_{y=y_s} + \lambda \frac{\partial T_2(y, \tau)}{\partial y}\bigg|_{y=y_s}\,, \quad \bigg| T_1(y, \tau) = \Delta T(y, \tau) + T_2(y, \tau) \bigg|_{y=y_s}\,. \tag{43}$$

Differentiating (2), (33), and (42) and substituting into (43), we find

$$A = \exp\left(\frac{y_s}{2}(\alpha_v - h)\right). \tag{44}$$

Substituting (44) into (42), subject to (2), we obtain an expression to calculate the temperature distribution in the region of $y \geq y_s$ with wave disturbance of the temperature field:

$$\begin{aligned} \tilde{T}_2(y, \tau) = T_0 + (\overline{T}_w - T_0) \exp\left(-\alpha_v (y - y_s) - \frac{\overline{V}_\infty}{a} S(\tau)\right) \operatorname{erf}\left(\frac{y - y_s}{2\sqrt{a\tau}}\right) \\ + (\overline{T}_w - T_0) \exp\left(-\frac{\overline{V}_\infty}{a}[y - S(\tau)]\right)\left[1 - \operatorname{erf}\left(\frac{y - y_s}{2\sqrt{a\tau}}\right)\right]. \end{aligned} \tag{45}$$

The solution (45) satisfies exactly the boundary conditions and with a small residual — the heat-conduction equation.

To obtain the additional term which takes into account the wave disturbance on surface destruction without mass entrainment, in (42) we exclude the parameters related to the surface displacement, then we find

$$\Delta T(y, \tau) + B\,(T_w - T_0) \exp(-\alpha_T y) \operatorname{erf}\left(\frac{y}{2\sqrt{a\tau}}\right). \tag{46}$$

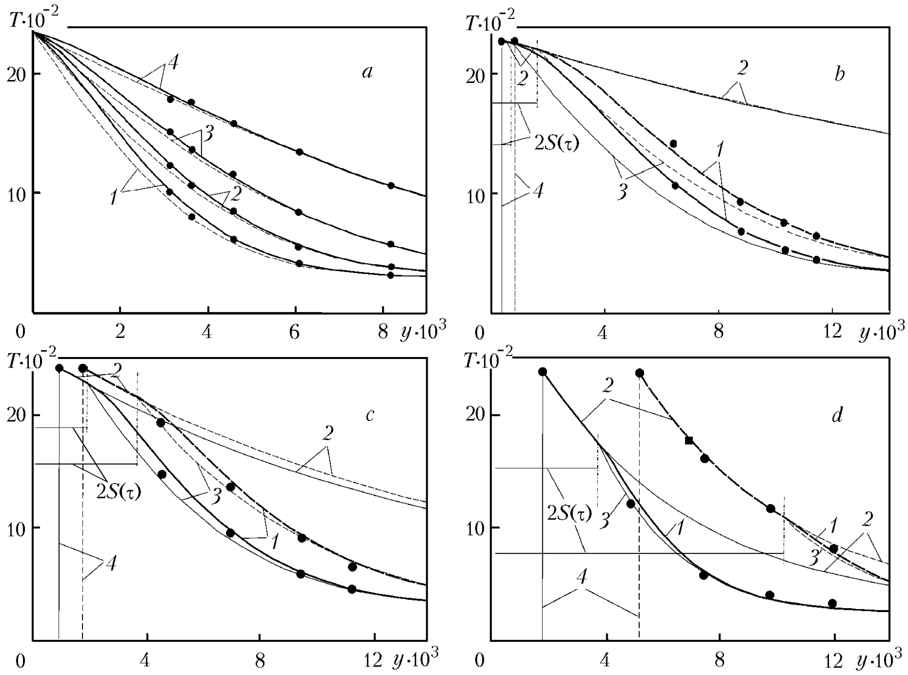

Fig. 5. Dynamics of heating of QGC specimens on change in the rate of surface destruction from ~0 to $0.11{\cdot}10^{-6}$ m/sec: a) $V_\infty \approx 0$ m/sec, $T_w = 2350$ K, $a = 0.6{\cdot}10^{-6}$ m^2/sec; solid lines — calculation by Eq. (47); dashed liens — by Eq. (36); 1–4) times of heating: 6, 13, 30, and 70 sec, $\tau_T = 13$ sec; b) $V_\infty = 0.02{\cdot}10^{-3}$ m/sec, $T_w = 2350$ K, $a = 0.65{\cdot}10^{-6}$ m^2/sec; solid lines — at the 30th second of heating, dashed lines — at the 50th second of heating; c) $V_\infty = 0.05{\cdot}10^{-3}$ m/sec, $T_w = 2400$ K, $a = 0.65{\cdot}10^{-6}$ m^2/sec; solid lines — at the 30th second of heating, dashed lines — at the 50th second of heating; d) $V_\infty = 0.11{\cdot}10^{-3}$ m/sec, $T_w = 2390$ K, $a = 0.6{\cdot}10^{-6}$ m^2/sec; solid lines — at the 20th second of heating, dashed lines — at the 50th second of heating; 1) calculation by Eq. (1) at $S(\tau) < y \le y_s$ and by Eq. (45) at $y \ge y_s$; 2) by Eq. (1); 3) by Eq. (2); 4) position of the heated surface; dots — readings of thermocouples; squares — the isotherm corresponding to the change in the color of alloyed QGC ($T^* = 1800$ K). T, K; y, m.

The coefficient K, which allows one to evaluate the zone of propagation of the wave disturbance of the temperature field by (40), and B are determined from the condition of the best agreement between the calculated and experimental results: $K = 1.6$, $B = 0.5$. Then the equation for calculation of the temperature distribution with allowance for wave disturbance in the case of surface melting without entrainment of the melt film takes the form

$$\tilde{T}(y, \tau) = T_0 + \frac{1}{2}(\bar{T}_w - T_0)\exp(-\alpha_T y)\,\mathrm{erf}\left(\frac{y}{2\sqrt{a\tau}}\right) + (\bar{T}_w - T_0)\,\mathrm{erfc}\left(\frac{y}{2\sqrt{a\tau}}\right). \tag{47}$$

It should be noted that the coefficient B may change from 0 to 1 and the value found lies at the middle of this range.

It is seen from Fig. 5 that in the range of mass entrainment velocities from ~0 to $0.11{\cdot}10^{-3}$ m/sec the proposed dependences (45) and (47) are in good agreement with experimental data and allow one to consider the dynamics of heating with thermal destruction of the material surface in a range sufficient for practice. It follows from Fig. 5a that in the case of surface melting without entrainment of the melt film the greatest deviation of the temperature field from the self-similar solution (36) is observed at heating times close to τ_T. On increase in the time of heating the calculations by (36) and (47) become closer, that is, we may assume that the wave disturbance damps out. This agrees well with experimental results.

In the case of mass entrainment from the surface, a stationary regime of heating is established sooner or later virtually for any isotherm of the temperature field. As is seen from Fig. 1, this process gradually propagates from the region of high temperatures into the region of low ones. The time of the establishment of a stationary temperature profile (1) is determined by both the parameter α_v, which determines the wave disturbance, and by the rate of heat transfer by the surface $\overline{V}_\infty/a$. From Eq. (45) it follows that at small velocities of entrainment the greatest effect on this process is exerted by wave disturbance (Fig. 5b), and the temperature field differs fairly strongly from the basic solution (2). At mass entrainment velocities exceeding $0.04 \cdot 10^{-3}$ m/sec the influence of the parameter α_v becomes insignificant, and the dependence (45) differs little from solution (2) (Fig. 5c). With a further increase in the entrainment velocity (Fig. 5d) the temperature distribution rapidly acquires a stationary (exponential) character (1).

Thus, the computational-experimental investigation of the dynamics of heating of an alloyed quartz glass ceramics confirms the wave disturbance (found in [4]) of the temperature field which may occur in a low-thermal-conductivity solid exposed to a heat loading that exceeds the level needed for heating without destruction of its surface.

NOTATION

a, thermal diffusivity, m^2/sec; $A(k)$ and $B(k)$, unknown functions of the integral argument k; c, heat capacity, kJ/(kg·K); C_1, constant; $C(q)$, unknown function of the integral argument q; d_0, parameter of nonstationary mass entrainment, m; D, symbol of the operation of differentiation of the function $T_{sh}(y_{sh}, k)$ with respect to the time argument τ in the field of transforms; D_y^2, symbol of operation of double differentiation of the function $T(y, \tau)$ with respect to the variable y in the field of transforms; $E(k)$, transform of the function exp $(-\overline{V}_\infty^2\tau/a)$; $E(q)$, transform of the function exp $(-hy)$; F_1, F_2, transforms of the function f; $H(T_w)$, heat content of the material at the surface temperature, kJ/(kg·K); H_1, H_2, given positive constants; H_y, positive scale constant for the variable y; H_τ, positive scale constant for the time argument τ; k_1 and k_2, integral arguments in the field of transforms with respect to the variables x_1 and x_2, respectively; k and q, integral arguments in the field of transforms with respect to the variables τ and y, respectively; K, coefficient characterizing the velocity of movement of an isotherm; K_{T_d}, constant of thermal destruction; n, positive integer; p, Laplace operator; q_c, heat flux to the cold surface of the calorimeter, kW/m^2; $S(\tau)$, $S(\tau_v)$, and $S(\tau_\delta)$, linear entrainment from the surface and its values at the instants of time τ_v and τ_δ, m; T, temperature, K; $T(y, \tau)$, temperature distribution function, K; $T(q, \tau)$, differential representation of function $T(y, \tau)$; $T_{sh}(y_{sh}, \tau)$, shifted variable in temperature, K; $T_{sh}(y_{sh}, k)$, differential representation of function $T_{sh}(y_{sh}, \tau)$; $T_1(y, \tau)$ and $T_2(y, \tau)$, temperature distribution over stationary and nonstationary sections of heating; K; $\tilde{T}(y, \tau)$ and $\tilde{T}_2(y, \tau)$, temperature distribution over nonstationary sections of heating with allowance for wave disturbance of temperature field without entrainment of mass from the surface and with entrainment, respectively, K; T_0, temperature of unheated material, K; T_w and $\overline{T}_w$, temperature of the heated surface and its steady-state value, K; T^*, temperature of an isotherm, K; T_d, temperature of the beginning of destruction (melting) of the material surface, K; $U(k_1, x_2)$ and $U(x_1, k_2)$, differential spectra of the function $v(x_1, x_2)$ with respect to variables x_1 and x_2, respectively; $V(k_1, x_2)$ and $V(x_1, k_2)$, differential spectra of the function $v_1(x_1, x_2)$ with respect to variables x_1 and x_2, respectively; V_∞ and $\overline{V}_\infty$, velocity of linear entrainment of mass and its stationary value, m/sec; y, coordinate reckoned from the original surface, m; y', coordinate reckoned from the mobile surface, m; y_{sh}, variable shifted over the coordinate, m; y_s, coordinate of the lower boundary of a stationary heated layer from the original surface, m; α_T and α_v, parameters characterizing wave disturbance of the temperature field without entrainment of mass from the surface and with its entrainment; β and ψ, continuous functions determined on the boundary surface Γ; δ_T, stationary value of the depth of the heated layer, m; Δ^*, total thickness of the heated and entrained layers, m; θ^*, dimensionless temperature of the isotherm; λ, thermal conductivity, W/(m·K); ρ, density, kg/m^3; $\sigma(k)$ and $\sigma(q)$, transforms of the constant single function over the time argument τ and variable y, respectively; τ, time of heating, sec; τ_T, τ_v, and τ_δ, time of establishment of stationary values of the surface temperature, velocity of mass entrainment, and heated layer thickness, sec; τ_ξ, a segment cut by linear dependence (39) on the abscissa axis ($\tau_\xi \approx 1$ sec); τ_d, time of the beginning of destruction (melting) of the surface, sec; Γ, boundary surface. Subscripts: 0, unheated material, impermeable surface; 1 and 2, stationary and nonstationary sections of the temperature field; s, lower boundary of the stationary heated layer (s-like); T, temperature; v, velocity; w, conditions on the wall; δ, heated layer; θ^*, dimensionless temperature of the isotherm; c, calorimetric; d, destruction; sh, shift; ∞, conditions at infinity.

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